Lecture Notes
MAT-101(Unit 1)

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Chapter 1

Series

Some Results on Sequences:

(1) $\lim_{n\to\infty} x^n = 0 \;\; (x < 1)$

(2) $\lim_{n\to\infty} nx^n = 0 \;\; (x < 1)$

(3) $\lim_{n\to\infty} x^n = \infty \;\; (x > 1)$

(4) $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

(5) $\lim_{n\to\infty} \frac{\log n}{n} = 0$

(6) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

(7) $\lim_{n\to\infty} (n)^{1/n} = 1$

(8) $\lim_{n\to\infty} (n!)^{1/n} = \infty$

(9) $\lim_{n\to\infty} \left[\frac{(n!)}{n}\right]^{1/n} = \frac{1}{e}$

(10) $\lim_{n\to\infty} (n)^n = \infty$

(11) $\lim_{n\to\infty} \frac{1}{n^n} = 0$

(12) $\lim_{x\to 0} \left(\frac{a^x - 1}{x}\right) = \ln a$

(13) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(14) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(15) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(16) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

A ***series*** is sum of sequence. Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a sequence. Then

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n$$

is called the series corresponding to the sequence $\{a_n\}$. For example, if $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is a sequence, then the corresponding series is $1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$.

Sequence of Partial Sum

Let $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ be a series. Consider

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\ldots\ldots \\ s_n &= a_1 + a_2 + a_3 + \ldots + a_n. \end{aligned}$$

Then $s_1, s_2, s_3 \ldots, s_n, \ldots$ is a sequence of real numbers. Note that for each $n \in \mathbb{N}$, s_n is the sum of first n terms of the series $\sum\limits_{n=1}^{\infty} a_n$. Therefore $\{s_n\}$ is called the ***sequence of partial sums*** of the series $\sum\limits_{n=1}^{\infty} a_n$.

Now, when we write $\sum a_n$ it means n is running from $n = 1$ to ∞ if that is not specified separately.

Nature of the Series

Consider a series $\sum\limits_{n=1}^{\infty} a_n$ and $\{s_n\}$ the corresponding sequence of partial sums. Then following three cases arise:

- If s_n tends to a finite number as $n \to \infty$, then series $\sum a_n$ is said to be *convergent.*
- If s_n tends to infinity as $n \to \infty$, then series $\sum a_n$ is said to be *divergent.*
- If s_n does not tend to a unique limit finite or infinite, then series $\sum a_n$ is said to be *oscillatory.*

Properties of Infinite Series

- If we add or remove finitely many terms in a series, then a convergent series remains convergent and a divergent series remains divergent.
- If we multiply each term of the series by a non-zero constant, then a convergent series remains convergent and a divergent series remains divergent.
- If $\sum a_n$ and $\sum b_n$ are convergent, then $\sum a_n \pm \sum b_n = \sum(a_n \pm b_n)$ is also convergent.

0.1. Definition (Geometric Series).

A series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots \quad (a, r \in \mathbb{R} \setminus \{0\})$$

is called the *geometric series.* The series is

(1) convergent if $|r| < 1$; (2) divergent if $r \geq 1$; (3) oscillatory if $r \leq -1$.

0.2. Example. Examine the convergence of the series:

(1) $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{3^4} + \ldots$

(2) $2 + 2^2 + 2^3 + 2^4 + \ldots$

Solution:(1)

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{3^4} + \ldots = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} \ldots\right)$$
$$= \sum a_n + \sum b_n \text{ (say)}$$

Now, $\sum a_n$ is a G.P. with common ratio $r = \frac{1}{2} < 1$, therefore $\sum a_n$ is convergent; similarly $\sum b_n$ is a G.P. with common ratio $r = \frac{1}{3^2} < 1$, therefore $\sum b_n$ is convergent. Thus, the given series $\sum(a_n + b_n)$ is also convergent.

(2) $2 + 2^2 + 2^3 + 2^4 + \ldots = \sum a_n$ (say). Which is a G.P. with common ratio $r = 2 > 1$, Thus, the series is divergent.

0.3. Example. Test the convergence or divergence of the following series:

(1) $\sum\limits_{n=0}^{\infty} \frac{2^n - 1}{3^n}$ (2) $\sum\limits_{n=1}^{\infty} \frac{4^n + 5^n}{6^n}$

Solution:(1)

$$\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n} = \sum_{n=0}^{\infty} \frac{2^n}{3^n} - \frac{1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \sum a_n - \sum b_n \text{ (say).}$$

Now, $\sum a_n$ is a G.P. with common ratio $r = \frac{2}{3} < 1$, therefore $\sum a_n$ is convergent; similarly, $\sum b_n$ is a G.P. with common ratio $r = \frac{1}{3} < 1$, therefore $\sum b_n$ is also convergent. Thus, the given series $\sum(a_n - b_n)$ is also convergent.

(2)

$$\sum_{n=1}^{\infty} \frac{4^n + 5^n}{6^n} = \sum_{n=1}^{\infty} \frac{4^n}{6^n} + \frac{5^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n = \sum a_n + \sum b_n \text{ (say).}$$

Now, $\sum a_n$ is a G.P. with common ratio $r = \frac{2}{3} < 1$, therefore $\sum a_n$ is convergent; similarly, $\sum b_n$ is a G.P. with common ratio $r = \frac{5}{6} < 1$, therefore $\sum b_n$ is also convergent. Thus, the given series $\sum(a_n + b_n)$ is also convergent.

0.4. Definition (Positive Term Series).

If all the terms after some finitely many terms of an infinite series are positive then such a series is called *positive term series*.
e.g. $-7 + 8 - 3 - 5 + 9 - 32 + \underbrace{2 + 3 + 5 + 34 + \ldots}_{\text{positive terms}}$ is a positive term series.

0.5. Theorem. *If $\sum a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.*

To find the nature of the series we have to find the sequence of partial sums $\{s_n\}$ of the series, but it is not possible to find $\{s_n\}$ for every series, and sometime it is difficult also. In this case the contrapositive statement of the above theorem, known as *Cauchy's fundamental test for divergence,* may be helpful to check for divergence of the series.

0.6. Theorem (Cauchy's Fundamental Test for Divergence).

$$\textit{If } \lim_{n\to\infty} a_n \neq 0, \textit{ then } \sum a_n \textit{ is divergent.}$$

0.7. Example. Discuss the convergence of the series:

$$\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots+\sqrt{\frac{n}{2n+2}}+\ldots$$

Solution: $\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\ldots+\sqrt{\frac{n}{2n+2}}+\ldots=\sum a_n$ (say). Then,

$$\lim_{n\to\infty} a_n=\lim_{n\to\infty}\sqrt{\frac{n}{2n+2}}=\lim_{n\to\infty}\sqrt{\frac{1}{2+\frac{2}{n}}}=\frac{1}{\sqrt{2}}\neq 0.$$

Therefore, $\sum a_n$ is not convergent.

p-Series

0.8. Theorem. *The series* $\sum \frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ *is*

(1) *convergent if* $p>1$;

(2) *divergent if* $p\leq 1$.

0.9. Example. Examine the convergence of the series:

(1) $1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\frac{1}{25^{2/3}}+\ldots$

(2) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots$

(3) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots$

Solution: (1)

$$1+\frac{1}{4^{2/3}}+\frac{1}{9^{2/3}}+\frac{1}{16^{2/3}}+\frac{1}{25^{2/3}}+\ldots=1+\frac{1}{2^{4/3}}+\frac{1}{3^{4/3}}+\frac{1}{4^{4/3}}+\frac{1}{5^{4/3}}+\ldots$$
$$=\sum\frac{1}{n^{4/3}}=\approx\frac{1}{n^p};$$

here $p=\frac{4}{3}>1$. By p-series test, the given series is convergent.

(2)

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots=\sum\frac{1}{n}\approx\sum\frac{1}{n^p};$$

here $p=1\leq 1$. By p-series test, the given series is divergent.

(3)

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots=\sum\frac{1}{\sqrt{n}}\approx\sum\frac{1}{n^p};$$

here $p=\frac{1}{2}\leq 1$. By p-series test, the given series is divergent.

Comparison Test

0.10. Theorem. *Let* $\sum a_n$ *and* $\sum b_n$ *be two positive term series such that*

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L;$$

then

(1) *if* L *is non-zero and finite, then* $\sum a_n$ *and* $\sum b_n$ *converge or diverge together;*

(2) *if* $L=0$, *then* $\sum a_n$ *is convergent if* $\sum b_n$ *is convergent;*

(3) *if* $L=\infty$, *then* $\sum a_n$ *is divergent if* $\sum b_n$ *is divergent.*

0.11. Example. Examine the convergence of the series:

(1) $\sum_{n=1}^{\infty} \dfrac{\sqrt{n^2-1}}{n^4+1}$

(2) $\sum_{n=1}^{\infty} \sqrt{n^2+1}-n$

(3) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots$

(4) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

(5) $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$

(6) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

Solution:(1) $\sum_{n=1}^{\infty} \dfrac{\sqrt{n^2-1}}{n^4+1} = \sum a_n$. Here $a_n = \dfrac{\sqrt{n^2-1}}{n^4+1}$

$$a_n = \frac{\sqrt{n^2-1}}{n^4+1} = \frac{n\sqrt{1-\frac{1}{n^2}}}{n^4(1+\frac{1}{n^4})} = \frac{\sqrt{1-\frac{1}{n^2}}}{n^3(1+\frac{1}{n^4})}$$

Take $b_n = \frac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{1-\frac{1}{n^2}}}{1+\frac{1}{n^4}} = \frac{1}{1} = 1 \neq 0 \text{ and finite.}$$

Therefore by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n^3} = \sum \frac{1}{n^p}$ with $p=3$, so $\sum b_n$ is convergent. Hence $\sum a_n$ is convergent.

(2) $\sum_{n=1}^{\infty} \sqrt{n^2+1}-n = \sum a_n$. Here $a_n = \sqrt{n^2+1}-n$

$$a_n = \sqrt{n^2+1}-n = \sqrt{n^2+1}-n \times \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$
$$= \frac{1}{\sqrt{n^2+1}+n} = \frac{1}{n(\sqrt{1+\frac{1}{n^2}}+1)};$$

Take $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2} \neq 0 \text{ and finite.}$$

Therefore by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n} = \sum \frac{1}{n^p}$ with $p=1$, so $\sum b_n$ is divergent. Hence $\sum a_n$ is divergent.

(3) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\dots = \sum a_n$. Take $a_n = \frac{n^n}{(n+1)^{n+1}}$ (**We omit the first term because nature of the series is not affected by omitting finitely many terms**.) We have

$$a_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}(1+\frac{1}{n})^{n+1}} = \frac{1}{n(1+\frac{1}{n})^{n+1}};$$

Take $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^{n+1}} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n}\frac{1}{(1+\frac{1}{n})} = \frac{1}{e}\frac{1}{1} = \frac{1}{e} \neq 0 \text{ and finite.}$$

Therefore by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n} = \sum \frac{1}{n^p}$ with $p = 1$, so $\sum b_n$ is divergent. Hence $\sum a_n$ is divergent.

(4) $\sum\limits_{n=1}^{\infty} \sin \frac{1}{n} = \sum a_n$. Here $a_n = \sin \frac{1}{n}$. Take $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{\frac{1}{n}\to 0} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \neq 0 \text{ and finite.}$$

Therefore by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n} = \sum \frac{1}{n^p}$ with $p = 1$, so $\sum b_n$ is divergent. Hence $\sum a_n$ is divergent.

(5) Try your self (Hint: Compare with $b_n = \frac{1}{n^2}$; convergent)

(6) $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} = \sum a_n$. Here $a_n = \frac{1}{n^{1+\frac{1}{n}}}$

$$a_n = \frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n \cdot n^{\frac{1}{n}}}$$

Take $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \frac{1}{1} = 1 \neq 0 \text{ and finite.}$$

Therefore by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n} = \sum \frac{1}{n^p}$ with $p = 1$, so $\sum b_n$ is divergent. Hence $\sum a_n$ is divergent.

0.12. Example. Discuss the convergence of the following series:

(1) $\sum\limits_{n=1}^{\infty} \dfrac{1}{1 + 2^2 + 3^2 + \ldots + n^2}$

(2) $\sum\limits_{n=1}^{\infty} \tan^{-1} \left(\dfrac{1}{n^2 + n + 1} \right)$

(3) $\sum\limits_{n=1}^{\infty} \dfrac{2n^2 + 3n}{5 + n^5}$

(4) $\sum\limits_{n=1}^{\infty} \dfrac{n^p}{\sqrt{n+1} + \sqrt{n}}$

Solution: (1) $\sum\limits_{n=1}^{\infty} \dfrac{1}{1 + 2^2 + 3^2 + \ldots + n^2} = \sum a_n$. Here,

$$a_n = \frac{1}{1 + 2^2 + 3^2 + \ldots + n^2} = \frac{1}{\frac{n(n+1)(2n+1)}{6}} = \frac{6}{n(n+1)(2n+1)} = \frac{6}{n^3(1+\frac{1}{n})(2+\frac{1}{n})}.$$

Take $b_n = \frac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6}{(1+\frac{1}{n})(2+\frac{1}{n})} = \frac{6}{3} = 2 \neq 0 \text{ and finite.}$$

Therefore, by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n^3} = \sum \frac{1}{n^p}$ with $p = 3$, so $\sum b_n$ is convergent. Hence $\sum a_n$ is convergent.

(2) $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \sum a_n$. Here, $a_n = \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$. Take $b_n = \frac{1}{n^2+n+1}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan^{-1}\left(\frac{1}{n^2+n+1}\right)}{\left(\frac{1}{n^2+n+1}\right)} = \lim_{(n^2+n+1)\to\infty} \frac{\tan^{-1}\left(\frac{1}{n^2+n+1}\right)}{\left(\frac{1}{n^2+n+1}\right)}$$

$$= \lim_{\frac{1}{n^2+n+1}\to 0} \frac{\tan^{-1}\left(\frac{1}{n^2+n+1}\right)}{\left(\frac{1}{n^2+n+1}\right)} = 1 \neq 0 \text{ and finite.}$$

Therefore, by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. Now we cheek wether $\sum b_n$ is convergent or divergent.

$$\sum b_n = \sum \frac{1}{n^2+n+1} = \sum \frac{1}{n^2(1+\frac{1}{n}+\frac{1}{n^2})}.$$

Take $c_n = \frac{1}{n^2}$, then

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} = 1 \neq 0 \text{ and finite.}$$

Therefore, by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum c_n = \sum \frac{1}{n^2} = \sum \frac{1}{n^p}$ with $p = 2$, so $\sum c_n$ is convergent. Hence $\sum b_n$ is convergent, and therefore $\sum a_n$ is also convergent.

(3) $\sum_{n=1}^{\infty} \frac{2n^2+3n}{5+n^5} = \sum a_n$. Here,

$$a_n = \frac{2n^2+3n}{5+n^5} = \frac{n^2(2+\frac{3}{n})}{n^5(1+\frac{5}{n^5})} = \frac{2+\frac{3}{n}}{n^3(1+\frac{5}{n^5})}.$$

Take $b_n = \frac{1}{n^3}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2+\frac{3}{n}}{1+\frac{5}{n^5}} = 2 \neq 0 \text{ and finite.}$$

Therefore, by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n^3} = \sum \frac{1}{n^p}$ with $p = 3$, so $\sum b_n$ is convergent. Hence, $\sum a_n$ is convergent.

(4) $\sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1}+\sqrt{n}} = \sum a_n$. Here,

$$a_n = \frac{n^p}{\sqrt{n+1}+\sqrt{n}} = \frac{n^p}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)} = \frac{1}{n^{\frac{1}{2}-p}\left(\sqrt{1+\frac{1}{n}}+1\right)}.$$

Take $b_n = \frac{1}{n^{\frac{1}{2}-p}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = 2 \neq 0 \text{ and finite.}$$

Therefore, by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \dfrac{1}{n^{\frac{1}{2}-p}} = \sum \frac{1}{n^q}$ with $q = \frac{1}{2} - p$. Now $\sum b_n$ is convergent if $\frac{1}{2} - p = q > 1$,i.e, $p < \frac{1}{2}$. Hence, $\sum a_n$ is convergent for $p < \frac{1}{2}$ and divergent if $p \geq \frac{1}{2}$.

0.13. Example. For which values of p does the series $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^p}$ is convergent.

$\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^p} = \sum a_n$. Here,

$$a_n = \frac{n+1}{n^p} = \frac{n\left(1+\frac{1}{n}\right)}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}.$$

Take $b_n = \dfrac{1}{n^{p-1}}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 \neq 0 \text{ and finite.}$$

Therefore, by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \dfrac{1}{n^{p-1}} = \sum \frac{1}{n^q}$ with $q = p - 1$. Now $\sum b_n$ is convergent if $p - 1 = q > 1$, i.e, $p > 2$. Hence, $\sum a_n$ is convergent for $p > 2$ and divergent if $p \leq 2$.

D'Alembert's Ratio Test

0.14. Theorem. *If $\sum a_n$ is a positive term series such that*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = L;$$

then

(1) *$\sum a_n$ is convergent if $L > 1$;*
(2) *$\sum a_n$ is divergent if $L < 1$;*
(3) *Test fails if $L = 1$.*

0.15. Example. Examine the convergence of the series:

(1) $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$

(2) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{3^n \cdot n^2}, \quad x > 0$

(3) $\sum\limits_{n=1}^{\infty} \dfrac{n2^n(n+1)!}{3^n n!}$

(4) $\sum\limits_{n=1}^{\infty} \dfrac{n}{e^{-n}}$

(5) $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dots$

(6) $\sum\limits_{n=1}^{\infty} \dfrac{n^3+2}{2^n+2}$

Solution: (1) $\sum\limits_{n=1}^{\infty} \dfrac{n!}{n^n}$, here $a_n = \dfrac{n!}{n^n}$. Thus,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{n^{n+1}\left(1+\frac{1}{n+1}\right)^{n+1}}{(n+1)\cdot n!}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \left(1 + \frac{1}{n+1}\right)^{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \cdot \left(1 + \frac{1}{n+1}\right)^{n+1} = 1 \cdot e = e > 1.$$

Therefore, by D'Alembert's ratio test, $\sum a_n$ is convergent.

(2) $\sum_{n=1}^{\infty} \frac{x^n}{3^n \cdot n^2}$, here $a_n = \frac{x^n}{3^n \cdot n^2}$. Thus,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{x^n}{3^n \cdot n^2} \cdot \frac{3^{n+1} \cdot (n+1)^2}{x^{n+1}} = \lim_{n\to\infty} \frac{3 \cdot (n+1)^2}{x \cdot n^2} = \lim_{n\to\infty} \frac{3 \cdot (1+\frac{1}{n})^2}{x} = \frac{3}{x}.$$

Therefore, by D'Alembert's ratio test , $\sum a_n$:
convergent if $\frac{3}{x} > 1$; divergent if $\frac{3}{x} < 1$ and test fails if $\frac{3}{x} = 1$; i.e. convergent if $x < 3$; divergent if $x > 3$ and test fails if $x = 3$.
For $x = 3$, $\sum_{n=1}^{\infty} \frac{x^n}{3^n \cdot n^2} = \sum_{n=1}^{\infty} \frac{3^n}{3^n \cdot n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$; which is convergent by p- series test. Thus, given series is convergent if $x \le 3$ and divergent if $x > 3$.

(3) $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$, here $a_n = \frac{n2^n(n+1)!}{3^n n!}$. Thus,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n \cdot 2^n \cdot (n+1)!}{3^n \cdot n!} \cdot \frac{3^{n+1} \cdot (n+1)!}{(n+1) \cdot 2^{n+1} \cdot (n+2)!} = \lim_{n\to\infty} \frac{3 \cdot n}{2 \cdot (n+2)}$$
$$= \lim_{n\to\infty} \frac{3}{2} \cdot \frac{1}{1+\frac{2}{n}} = \frac{3}{2} > 1.$$

Therefore, by D'Alembert's ratio test, $\sum a_n$ is convergent.

(4) $\sum_{n=1}^{\infty} \frac{n}{e^{-n}}$, here $a_n = \frac{n}{e^{-n}}$. Thus,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n}{e^{-n}} \cdot \frac{e^{-(n+1)}}{n+1} = \lim_{n\to\infty} \frac{1}{e} \cdot \frac{1}{1+\frac{1}{n}} = \frac{1}{e} < 1.$$

Therefore, by D'Alembert's ratio test, $\sum a_n$ is divergent.

(5) Here, $a_n = \frac{n}{(n+1)!}$. Thus,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n}{(n+1)!} \cdot \frac{(n+2)!}{n+1} = \lim_{n\to\infty} \frac{n(n+2)}{(n+1)} = \lim_{n\to\infty} n\left(1 + \frac{1}{n+1}\right) = \infty > 1.$$

Therefore, by D'Alembert's ratio test, $\sum a_n$ is convergent.

(6) Here, $a_n = \frac{n^3+2}{2^n+2}$. Thus,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n^3+2}{2^n+2} \cdot \frac{2^{n+1}+2}{(n+1)^3+2} = \lim_{n\to\infty} \frac{n^3\left(1+\frac{2}{n^3}\right)}{2^n\left(1+\frac{2}{2^n}\right)} \cdot \frac{2^{n+1}\left(1+\frac{2}{2^{n+1}}\right)}{n^3\left[\left(1+\frac{1}{n}\right)^3 + \frac{2}{n^3}\right]}$$
$$= \lim_{n\to\infty} \frac{1+0}{1+0} \cdot \frac{2(1+0)}{(1+0)^3+0} = 2 > 1.$$

Therefore, by D'Alembert's ratio test, $\sum a_n$ is convergent.

Cauchy's n^{th} Root Test

0.16. Theorem. *If $\sum a_n$ is a positive term series such that*

$$\lim_{n\to\infty} (a_n)^{1/n} = L;$$

then

(1) $\sum a_n$ *is convergent if* $L < 1$ *;*
(2) $\sum a_n$ *is divergent if* $L > 1$ *;*
(3) *Test fails if* $L = 1$.

0.17. Example. Examine the convergence of the following series:

(1) $\sum_{n=1}^{\infty} \frac{(n-\log n)^n}{2^n \cdot n^n}$
(2) $\frac{1}{2} + \frac{2}{3}x + (\frac{3}{4})^2 x^2 + (\frac{4}{5})^3 x^3 + \dots, x \geq 0.$

Solution: (1) $\sum_{n=1}^{\infty} \frac{(n-\log n)^n}{2^n \cdot n^n}$, here $a_n = \frac{(n-\log n)^n}{2^n \cdot n^n}$. Thus,

$$\begin{aligned}\lim_{n\to\infty} (a_n)^{\frac{1}{n}} &= \lim_{n\to\infty} \Big(\frac{(n-\log n)^n}{2^n \cdot n^n}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{2}\Big(\frac{n-\log n}{n}\Big) = \lim_{n\to\infty} \frac{1}{2}\Big(1 - \frac{\log n}{n}\Big) \\ &= \frac{1}{2}\Big(1 - \lim_{n\to\infty} \frac{\log n}{n}\Big) = \frac{1}{2}\Big(1 - \lim_{n\to\infty} \frac{\frac{1}{n}}{1}\Big) \quad \Big(\frac{\infty}{\infty} \text{ form; By L'Hopital rule}\Big) \\ &= \frac{1}{2}(1-0) = \frac{1}{2} < 1.\end{aligned}$$

Therefore, by Cauchy's root test the series is convergent.

(2) $\frac{1}{2} + \frac{2}{3}x + (\frac{3}{4})^2 x^2 + (\frac{4}{5})^3 x^3 + \dots$. Omit the first term, we have $\sum a_n = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n x^n$. Thus, $a_n = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n x^n$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{n+1}{n+2}\right)^n x^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n\ (1+\frac{1}{n})}{n\ (1+\frac{2}{n})}\, x = x.$$

Therefore, by Cauchy's root test the series is convergent if $x < 1$; divergent if $x > 1$ and test fails if $x = 1$.
For $x = 1$; $\sum a_n = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n$. Here $a_n = \left(\frac{n+1}{n+2}\right)^n$. But,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} = \frac{1}{e} \neq 0.$$

Therefore, the series is not convergent for $x = 1$. Thus, the series is convergent if $x < 1$ and divergent if $x \geq 1$.

Raabe's Test

0.18. Theorem. *If* $\sum a_n$ *is a positive term series such that*

$$\lim_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) = L;$$

then

(1) $\sum a_n$ *is convergent if* $L > 1$ *;*
(2) $\sum a_n$ *is divergent if* $L < 1$ *;*
(3) *Test fails if* $L = 1$.

Remark: The Raabe's test is used when D'Alembert's ratio test is failed and the ratio $\frac{a_n}{a_{n+1}}$ does not contains the number e.

0.19. Example. Examine the convergence of the series:

$$1 + \frac{3}{7} + \frac{3\cdot 6}{7\cdot 10} + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13} + \dots$$

Solution: Omitting the first term, we have

$$\frac{3}{7} + \frac{3\cdot 6}{7\cdot 10} + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13} + \dots = \sum_{n=1}^{\infty} \frac{3\cdot 6\dots(3n)}{7\cdot 10\dots(3n+4)} = \sum_{n=1}^{\infty} a_n$$

Here,

$$a_n = \frac{3\cdot 6\dots(3n)}{7\cdot 10\dots(3n+4)}.$$

Therefore,

$$a_{n+1} = \frac{3\cdot 6\dots(3(n+1))}{7\cdot 10\dots(3(n+1)+4)} = \frac{3\cdot 6\dots(3n)\cdot(3n+3)}{7\cdot 10\dots(3n+4)\cdot(3n+7)}.$$

Now

$$\frac{a_n}{a_{n+1}} = \frac{3\cdot 6\dots(3n)}{7\cdot 10\dots(3n+4)}\,\frac{7\cdot 10\dots(3n+4)\cdot(3n+7)}{3\cdot 6\dots(3n)\cdot(3n+3)} = \frac{3n+7}{3n+3};$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{3n+7}{3n+3} = \lim_{n\to\infty}\frac{1+\frac{7}{3n}}{1+\frac{3}{3n}} = 1.$$

D'Alembert's ratio test is failed. Now we will apply Raabe's test

$$n\Big(\frac{a_n}{a_{n+1}} - 1\Big) = n\Big(\frac{3n+7}{3n+3} - 1\Big) = n\Big(\frac{4}{3n+3}\Big) = \frac{4n}{3n+3};$$

$$\lim_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) = \lim_{n\to\infty}\frac{4n}{3n+3} = \lim_{n\to\infty}\frac{4}{3+\frac{3}{n}} = \frac{4}{3} > 1.$$

By Raabe's test, the series is convergent.

0.20. Example. Examine the convergence of the series:

$$\sum \frac{1\cdot 3\cdot 5\dots(2n-1)}{2\cdot 4\cdot 6\dots(2n)(2n+1)} x^{2n+1}$$

Solution: Here,

$$a_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)(2n+1)} x^{2n+1}.$$

Therefore,

$$a_{n+1} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \ldots (2n)(2n+2)(2n+3)} x^{2n+3}.$$

Now

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)\; x^{2n+1}}{2 \cdot 4 \cdot 6 \ldots (2n)(2n+1)} \; \frac{2 \cdot 4 \cdot 6 \ldots (2n)(2n+2)(2n+3)}{1 \cdot 3 \cdot 5 \ldots (2n-1)(2n+1)\; x^{2n+3}}; \\ &= \frac{(2n+2)(2n+3)}{(2n+1)^2 x^2} \\ \lim_{n\to\infty} \frac{a_n}{a_{n+1}} &= \lim_{n\to\infty} \frac{(2+2/n)(2+3/n)}{(2+1/n)^2 x^2} = \frac{1}{x^2}.\end{aligned}$$

By D'Alembert's ratio test, the series is

(1) convergent if $\frac{1}{x^2} > 1$ or $x^2 < 1$

(2) divergent if $\frac{1}{x^2} < 1$ or $x^2 > 1$

(3) Test fail if $\frac{1}{x^2} = 1$

Now we will apply Raabe's test for $x^2 = 1$. Then

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \frac{(2n+2)(2n+3)}{(2n+1)^2} \\ \Rightarrow\; n\Big(\frac{a_n}{a_{n+1}} - 1\Big) &= n\Big(\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\Big) = \frac{n(6n+5)}{(2n+1)^2} = \frac{(6+5/n)}{(2+1/n)^2}; \\ \Rightarrow\; \lim_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) &= \lim_{n\to\infty} \frac{(6+5/n)}{(2+1/n)^2} = \frac{3}{2} > 1.\end{aligned}$$

By Raabe's test, the series is convergent if $x^2 = 1$.
Hence the series is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

0.21. Example (H.W.)**.** Eximine the convergence of $1 + \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \ldots$
Answer: Divergent.

0.22. Example (H.W.)**.** Eximine the convergence of $\sum \frac{4 \cdot 7 \cdot 10 \ldots (3n+1)}{1 \cdot 2 \cdot 3 \ldots n} x^n$
Answer: convergent if $x < 1/3$ and divergent if $x \ge 1/3$.

Logarithmic Test

0.23. Theorem. *If $\sum a_n$ is a positive term series such that*

$$\lim_{n\to\infty}\left(n\log\frac{a_n}{a_{n+1}}\right) = L;$$

then

(1) $\sum a_n$ *is convergent if* $L > 1$;
(2) $\sum a_n$ *is divergent if* $L < 1$;
(3) *Test fails if* $L = 1$.

0.24. Example. Examine the convergence of the series:

$$\frac{1}{e} + \frac{2^2}{2!\cdot e^2} + \frac{3^3}{3!\cdot e^3} + \frac{4^4}{4!\cdot e^4} + \dots$$

Solution: We have,

$$\frac{1}{e} + \frac{2^2}{2!\cdot e^2} + \frac{3^3}{3!\cdot e^3} + \frac{4^4}{4!\cdot e^4} + \dots = \sum_{n=1}^{\infty}\frac{n^n}{n!\cdot e^n} = \sum_{n=1}^{\infty} a_n$$

Here, $a_n = \dfrac{n^n}{n!\cdot e^n}$. Thus,

$$\frac{a_n}{a_{n+1}} = \frac{n^n}{n!\cdot e^n}\,\frac{(n+1)!\cdot e^{n+1}}{(n+1)^{n+1}} = \frac{n^n\cdot(n+1)\cdot e}{(n+1)(n+1)^n} = \frac{e}{\left(\frac{n+1}{n}\right)^n} = \frac{e}{\left(1+\frac{1}{n}\right)^n};$$

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty}\frac{e}{\left(1+\frac{1}{n}\right)^n} = \frac{e}{e} = 1$$

D'Alembert's ratio test is failed. Since $\frac{a_n}{a_{n+1}}$ contains the term e, we will apply Logarithmic test.

$$\begin{aligned}
\lim_{n\to\infty}\left(n\log\frac{a_n}{a_{n+1}}\right) &= \lim_{n\to\infty} n\left(\log\frac{e}{(1+\frac{1}{n})^n}\right) = \lim_{n\to\infty} n\Big[(\log e - n\log(1+\frac{1}{n})\Big] \\
&= \lim_{n\to\infty} n\Big[1 - n(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots)\Big] \\
&= \lim_{n\to\infty} n\Big[\frac{1}{2n} - \frac{1}{3n^2} + \dots\Big] \\
&= \lim_{n\to\infty}\Big[\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \dots\Big] = \frac{1}{2} < 1
\end{aligned}$$

By Logarithmic test the series is divergent.
Remark: The Logarithmic test is used when D'Alembert's ratio test is fails and the ratio $\frac{a_n}{a_{n+1}}$ contains the number e.

Gauss's Test

0.25. Theorem. *If $\sum a_n$ is a positive term series such that*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\lambda}{n} + \frac{\alpha_n}{n^{1+\delta}}$$

where $\delta > 0$ and $\{\alpha_n\}$ is a bounded sequence, then

(1) *$\sum a_n$ is convergent if $\lambda > 1$;*
(2) *$\sum a_n$ is divergent if $\lambda \leq 1$.*

0.26. Example. Examine the convergence of the series:

$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

Solution: Here $a_n = \dfrac{1^2 \cdot 3^2 \cdot 5^2 \dots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2}$, therefore

$$a_{n+1} = \frac{1^2 \cdot 3^2 \cdot 5^2 \dots (2(n+1)-1)^2}{2^2 \cdot 4^2 \cdot 6^2 (2(n+1))^2} = \frac{1^2 \cdot 3^2 \cdot 5^2 \dots\dots (2n-1)^2 \cdot (2n+1)^2}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2 \cdot (2n+2)^2}$$

$$\frac{a_n}{a_{n+1}} = \frac{1^2 \cdot 3^2 \cdot 5^2 \dots (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2} \, \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2 \cdot (2n+2)^2}{1^2 \cdot 3^2 \cdot 5^2 \dots\dots (2n-1)^2 \cdot (2n+1)^2} = \frac{(2n+2)^2}{(2n+1)^2};$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+1)^2} = \lim_{n\to\infty} \frac{(1+2/2n)^2}{(1+1/2n)^2} = 1$$

D'Alembert's ratio test is failed. Raabe's test is also failed (cheek!!!), now we will apply Gauss's test.

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \frac{(1+2/2n)^2}{(1+1/2n)^2} = (1+2/2n)^2(1+1/2n)^{-2} = (1+1/n)^2(1+1/2n)^{-2} \\ &= \Big(1+\frac{2}{n}+\frac{1}{n^2}\Big)\Big(1-2\frac{1}{2n}+3\frac{1}{4n^2}-4\frac{1}{8n^3}+\dots\Big) \\ &= 1+\frac{1}{n}+\frac{1}{n^2}(1-2+\frac{3}{4})+\dots = 1+\frac{1}{n}+O\Big(\frac{1}{n^2}\Big)\end{aligned}$$

Comparing with $\frac{a_n}{a_{n+1}} = 1 + \frac{\lambda}{n} + O\Big(\frac{1}{n^2}\Big)$ we have $\lambda = 1$. Thus, by Gauss test the series is divergent.

Remark: Gauss test is applied when D'Alebert's ratio test fails and it is possible to express $\frac{a_n}{a_{n+1}}$ in powers of $\frac{1}{n}$. Generally Binomial Theorem is used to express $\frac{a_n}{a_{n+1}}$ in powers of $\frac{1}{n}$.

Cauchy's Integral Test

0.27. Theorem. *If for $x \geq 1$, $f(x)$ is a non-negative monotonically decreasing integrable function of x such that $f(n) = a_n$ for $n \in \mathbb{N}$, then*

(1) $\sum\limits_{n=1}^{\infty} a_n$ *is converge if* $\int_1^\infty f(x)\,dx$ *is finite;*

(2) $\sum\limits_{n=1}^{\infty} a_n$ *is diverge if* $\int_1^\infty f(x)\,dx$ *is infinite.*

0.28. Example. Test for convergence the series: $\sum\limits_{n=1}^{\infty} ne^{-n^2}$.

Solution: Here, $a_n = ne^{-n^2}$. Take $f(x) = xe^{-x^2}$, then $f(x) \geq 0$ for all x and $a_n = ne^{-n^2} = f(n)$. Since, $f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = (1-2x^2)e^{-x^2} < 0$ for all $x \geq 1$, $f(x)$ is decreasing. Thus, $f(x)$ is positive and decreasing.

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty}\int_1^n f(x)\,dx = \lim_{n\to\infty}\int_1^n xe^{-x^2}\,dx$$

$$\text{Let } -x^2 = t \Rightarrow -2x\,dx = dt \Rightarrow 2x\,dx = -\frac{dt}{2},$$

$$x \to 1 \Rightarrow t \to -1 \text{ and } x \to n \Rightarrow t \to -n^2.$$

$$= \lim_{n\to\infty} -\frac{1}{2}\int_{-1}^{-n^2} e^t\,dt = \lim_{n\to\infty} -\frac{1}{2}[e^t]_{-1}^{-n^2} = \lim_{n\to\infty} -\frac{1}{2}[e^{-n^2} - e^{-1}]$$

$$= \lim_{n\to\infty} \frac{1}{2}[e^{-1} - e^{-n^2}] = \frac{1}{2e} \text{ which is finite.}$$

$$\Rightarrow \int_1^\infty f(x)\,dx \text{ converges } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

0.29. Example. Discuss the convergence of the series: $\sum\limits_{n=1}^{\infty} \frac{1}{n(1+\log n)^2}$.

Solution: Here, $a_n = \frac{1}{n(1+\log n)^2}$. Take $f(x) = \frac{1}{x(1+\log x)^2}$, then $f(x) \geq 0$ for all $x \geq 1$ and $a_n = \frac{1}{n(1+\log n)^2} = f(n)$. Clearly, $f(x)$ is decreasing for all $x \geq 1$. Thus, $f(x)$ is positive and decreasing.

$$\int_1^\infty f(x)\,dx = \lim_{n\to\infty}\int_1^n f(x)\,dx = \lim_{n\to\infty}\int_1^n \frac{1}{x(1+\log x)^2}\,dx$$

$$= \lim_{n\to\infty}\int_1^n \frac{1}{x}(1+\log x)^{-2}\,dt = \lim_{n\to\infty}\left[\frac{(1+\log x)^{-1}}{-1}\right]_1^n$$

$$= \lim_{n\to\infty}[1 - \frac{1}{1+\log n}] = 1 \text{ which is finite.}$$

$\Rightarrow \int\limits_1^\infty f(x)\,dx$ converges $\Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges.

Alternating Series

A series of positive (or negative) terms either converge or diverge but never oscillate. A series having all terms negative also either converge or diverge but never oscillate, because taking the factor (-1) common from each term then the remaining series is of positive terms. Also a series with finitely many terms of one sign and the remaining terms of other sign either converge or diverge but never oscillates, as the nature of the series does not affected by omitting finitely many terms. If a series for which the limit of n^{th} term is not zero and having infinitely many positive and negative terms then such a series is called series of ***arbitrary terms***. By above tests we can not decide the nature of such series.

0.30. Definition.

A series with terms alternately positive and negative is called an ***alternating series***. For example,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

The general form of alternating series is given by

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + \dots = \sum_{n=1}^{\infty}(-1)^{n+1}a_n \quad (a_n > 0)$$

Leibnitz's Test on Alternating Series

0.31. Theorem. *The alternating series*

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + \dots \quad (a_n > 0)$$

converges if

(1) $a_n \geq a_{n+1}$ *for all* $n \in \mathbb{N}$ *and* (2) $\lim\limits_{n\to\infty} a_n = 0$.

Absolute and Conditional Convergence

0.32. Definition.

A series $\sum\limits_{n=1}^{\infty} a_n$ is said to be ***absolutely convergent*** if the series $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent.

0.33. Definition.

If $\sum\limits_{n=1}^{\infty} a_n$ is converges but $\sum\limits_{n=1}^{\infty} |a_n|$ diverges, then the series $\sum\limits_{n=1}^{\infty} a_n$ is called ***conditionally convergent***

Remark 1: Every absolutely convergent series is a convergent series but the converse is not true.

Remark 2: If $\sum\limits_{n=1}^{\infty} a_n$ is a series of positive terms, then $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} a_n$. Therefor the concepts of convergence and absolutely convergence are the same. Thus, any convergent series of positive terms is also absolutely convergent.

0.34. Example. Test the convergence of the series: $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$

Solution: $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots = \sum\limits_{n=0}^{\infty} (-1)^n \frac{2^n \cdot 5}{3^n} = \sum\limits_{n=0}^{\infty} (-1)^n a_n$. The given series is alternating series. Here

$$a_n = \frac{2^n \cdot 5}{3^n} > 0, \text{ for all } n \in \mathbb{N}, \text{ and } a_{n+1} = \frac{2^{n+1} \cdot 5}{3^{n+1}}.$$

Observe that

$$\begin{aligned} a_n - a_{n+1} &= \frac{2^n \cdot 5}{3^n} - \frac{2^{n+1} \cdot 5}{3^{n+1}} = 5\Big[\frac{2^n}{3^n} - \frac{2^{n+1}}{3^{n+1}}\Big] = 5\Big[\frac{3 \cdot 2^n - 2^{n+1}}{3^{n+1}}\Big] \\ &= 5\Big[\frac{2^n(3-2)}{3^{n+1}}\Big] = \frac{5 \cdot 2^n}{3^{n+1}} > 0 \end{aligned}$$

Thus $a_n > a_{n+1}$, for all $n \in \mathbb{N}$. Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 5\Big(\frac{2}{3}\Big)^n = 0$$

Thus, both the conditions of Leibnitz's test are holds. Therefore, by Leibnitz's test the $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

0.35. Example. Test the convergence and absolute convergence of the series:

(1) $\frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} - \frac{1}{4 \cdot 6} + \dots$

(2) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Solution: (1) The given series is alternating series.

$$\frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} - \frac{1}{4 \cdot 6} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+2)} = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n+1} a_n.$$

Here,

$$a_n = \frac{1}{n(n+2)} > 0 \text{ for all } n \in \mathbb{N} \text{ and } a_{n+1} = \frac{1}{(n+1)(n+3)}.$$

Observe that, $\frac{1}{n} > \frac{1}{n+1}$ and $\frac{1}{n+2} > \frac{1}{n+3}$ it follows that;

$$a_n = \frac{1}{n(n+2)} > \frac{1}{(n+1)(n+3)} = a_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Thus, $a_n > a_{n+1}$ for all $n \in \mathbb{N}$. Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+2)} = 0.$$

Thus, both the conditions of Leibnitz's test are hold. Therefore, by Leibnitz's test the $\sum_{n=1}^{\infty} b_n$ is convergent. Now,

$$\sum |b_n| = \sum |(-1)^{n+1} a_n| = \sum a_n = \sum \frac{1}{n(n+2)}.$$

This gives

$$|b_n| = \frac{1}{n(n+2)} = \frac{1}{n^2\left(1+\frac{2}{n}\right)}$$

Taking $u_n = \frac{1}{n^2}$, we have

$$\lim_{n\to\infty} \frac{|b_n|}{u_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)} = 1 \neq 0 \text{ and finite.}$$

Therefore by comparison test $\sum |b_n|$ and $\sum u_n$ converge or diverge together. But $\sum u_n = \sum \frac{1}{n^2} = \sum \frac{1}{n^p}$ with $p = 2$ is convergent. Hence, $\sum |b_n|$ is also convergent. Which implies that the given series is absolutely convergent.

(2) The given series is alternating series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n.$$

Here,

$$a_n = \frac{1}{\sqrt{n}} > 0 \text{ for all } n \in \mathbb{N}, \text{ and } a_{n+1} = \frac{1}{\sqrt{n+1}}.$$

Observe that,

$$a_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = a_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

Thus, $a_n > a_{n+1}$ for all $n \in \mathbb{N}$. Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Thus, both the conditions of Leibnitz's test are holds. Therefore, by Leibnitz's test the $\sum_{n=1}^{\infty} b_n$ is convergent. Now

$$\sum |b_n| = \sum |(-1)^{n+1} a_n| = \sum a_n = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^p},$$

with $p = \frac{1}{2} < 1$ is divergent. Hence $\sum |b_n|$ is divergent convergent. Thus, given series is conditionally convergent.

0.36. Example. Test the convergence of the following series:

(1) $1 - 2x + 3x^2 - 4x^3 + \ldots \infty, \quad (0 < x < 1)$

(2) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1} x^{n+1}$

(3) $\sum\limits_{n=0}^{\infty} n!(x-4)^n$

Solution: (1) Here, $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$.

$$\frac{|a_n|}{|a_{n+1}|} = \frac{n x^{n-1}}{(n+1)x^n} = \frac{1}{\left(\frac{n+1}{n}\right)} \frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)} \frac{1}{x}$$

Therefore,

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} \frac{1}{x} = \frac{1}{x}$$

Therefore, by D'Alembert's ratio test, $\sum |a_n|$ is convergent if $\dfrac{1}{x} > 1$ i.e., $x < 1$. Since absolutely convergent series is also convergent, $\sum a_n$ is convergent if $x < 1$.

(2) Here, $a_n = \dfrac{(-1)^n}{2n-1} x^{n+1}$ and $a_{n+1} = \dfrac{(-1)^{n+1}}{2n+1} x^{n+2}$.

$$\frac{|a_n|}{|a_{n+1}|} = \frac{|x|^{n+1}}{|2n-1|} \cdot \frac{|2n+1|}{|x|^{n+2}} = \frac{(2n+1)}{(2n-1)} \cdot \frac{1}{|x|} = \frac{(2+\frac{1}{n})}{(2-\frac{1}{n})} \cdot \frac{1}{|x|}.$$

Therefore,

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(2+\frac{1}{n})}{(2-\frac{1}{n})} \cdot \frac{1}{|x|} = \frac{1}{|x|}.$$

Therefore, by D'Alembert's ratio test, $\sum |a_n|$ is convergent if $\dfrac{1}{|x|} > 1$ i.e., $|x| < 1$ i.e., $-1 < x < 1$. Also $\sum |a_n|$ is divergent if $|x| > 1$. Test fails when $|x| = 1$.

When $x = 1$, $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{2n-1} \approx \sum\limits_{n=1}^{\infty} (-1)^n b_n$. It is an alternating series. Here

$$b_n = \frac{1}{2n-1}, \quad b_{n+1} = \frac{1}{2n+1}.$$

Clearly, $b_n > b_{n+1}$ for all n. Also $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{2n-1} = 0$. Therefore, by Leibnitz's test, $\sum a_n$ is convergent.

When $x = -1$, $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n(-1)^n}{2n-1} = \sum\limits_{n=1}^{\infty} \dfrac{1}{2n-1}$. Here $a_n = \dfrac{1}{2n-1} = \dfrac{1}{n\left(2-\frac{1}{n}\right)}$. Take $b_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0 \text{ and finite.}$$

Therefore, by comparison test $\sum a_n$ and $\sum b_n$ converge or diverge together. But $\sum b_n = \sum \frac{1}{n} = \sum \frac{1}{n^p}$ with $p = 1$, so $\sum b_n$ is divergent. Hence, $\sum a_n$ is also divergent.
Since absolutely convergent series is convergent, $\sum a_n$ is convergent for $-1 < x \leq 1$.

(3) Here, $a_n = n!(x-4)^n$ and $a_{n+1} = (n+1)!(x-4)^{n+1}$.

$$\frac{|a_n|}{|a_{n+1}|} = \frac{n! \cdot |x-4|^n}{(n+1)! \cdot |x-4|^{n+1}} = \frac{1}{n+1}\frac{1}{|x-4|}.$$

Therefore,

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{1}{n+1}\frac{1}{|x-4|} = \frac{1}{|x-4|}.$$

Therefore, by D'Alembert's ratio test, $\sum |a_n|$ is convergent if $\frac{1}{|x-4|} > 1$ i.e., $|x-4| < 1$ i.e., $-1 < x-4 < 1$ i.e., $3 < x < 5$. Also $\sum |a_n|$ is divergent if $|x-4| > 1$. Test fails when $|x-4| = 1$ i.e., $x - 4 = \pm 1$ i.e., $x = 5$ or $x = -3$.
When $x = 5$, $\sum a_n = \sum n!$. Here $a_n = n!$. Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n! \neq 0$$

$\sum a_n$ is not convergent.
When $x = -3$, $\sum a_n = \sum n!(-7)^n = \sum(-1)^n 7^n n! \approx \sum(-1)^n b_n$. Here $b_n = 7^n n!$. Now,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} 7^n n! \neq 0$$

By Lebnitz's test, $\sum a_n$ is not convergent.
Hence, $\sum a_n$ is convergent for $3 < x < 5$.

0.37. Definition. (Power Series)

A *power series* in powers of $(x-a)$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$$

$a_0, a_1, a_2, a_3, \dots$ are constants, called the coefficients of the series, a is a constant, called the center of the series and x is a variable.
If in particular $a = 0$, we obtain a power series in powers of x

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Familiar examples of the power series are Maclaurian series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad (|x| < 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Radius of Convergence

- The power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is always converges at $x = a$, because then all the terms except for the first, a_0, are zero. Such a series is of no particular interest.
- If there are further values of x for which the series is converges, these values form an interval, called the ***interval of convergence***. If this interval is finite, then it has the mid point a, so the interval is of the form
$$|x - a| < R$$
- The power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is converges for all x such that $|x-a| < R$ and diverges for all x such that $|x-a| > R$. The number R is called the ***radius of convergence*** of the power series The power series $\sum_{n=0}^{\infty} a_n(x-a)^n$.
- The radius of convergence, R, can be obtained from either of the formulas
$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \quad \text{or} \quad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$
provided these limits exists and are not zero. If they are infinite, then the power series converges only at the center a.

0.38. Example. Find the radius of convergence and interval of convergence of the series
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$
.

Solution: This is a power series in powers of x with $a_n = 1$. So that,
$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1}{1}\right| = 1$$
Thus $R = 1$. Hence the series converges for $|x| < 1$. Thus radius of convergence is 1 and interval of convergence is $(-1, 1)$.

0.39. Example. Find the radius of convergence and interval of convergence of the series
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
.

Solution: This is a power series in powers of x with $a_n = \frac{1}{n!}$. So that,
$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Thus $R = \infty$. Hence the series converges for x. Thus radius of convergence is ∞ and interval of convergence is $\mathbb{R}$.

0.40. Example. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} n! x^n$$

.

Solution: This is a power series in powers of x with $a_n = n!$. So that,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty}(n+1) = \infty.$$

Thus $R = 0$. Hence the series converges only at center $x = 0$. Such a series is useless.

0.41. Example. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n+2}$$

.

Solution: This is a power series in powers of x with $a_n = \dfrac{1}{n+2}$. So that,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+3}\right| = \lim_{n\to\infty}\frac{1+2/n}{1+3/n} = 1.$$

Thus $R = 1$. Hence the series converges for $|x| < 1$. Thus radius of convergence is 1 and interval of convergence is $(-1, 1)$.

0.42. Example. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

.

Solution: This is a power series in powers of x with $a_n = \dfrac{(-3)^n}{\sqrt{n+1}}$. So that,

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}}{\sqrt{n+2}}\,\frac{\sqrt{n+1}}{(-3)^n}\right| = \lim_{n\to\infty} 3\sqrt{\frac{n+1}{n+2}} = \lim_{n\to\infty} 3\sqrt{\frac{1+1/n}{1+2/n}} = 3$$

Thus $R = \frac{1}{3}$. Hence the series converges for $|x| < \frac{1}{3}$. Thus radius of convergence is $1/3$ and interval of convergence is $(-\frac{1}{3}, \frac{1}{3})$.

0.43. Example. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} x^{3n}$$

.

Solution: Here
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} x^{3n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} (x^3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} t^n$$
This is a power series in powers of $t = x^3$ with $a_n = \dfrac{(-1)^n}{8^n}$. So that,
$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{8^{n+1}} \frac{8^n}{(-1)^n}\right| = \lim_{n\to\infty} \left|\frac{1}{8}\right| = \frac{1}{8}.$$
Thus $R = 8$. Hence the series converges for
$$|t| < 8 \quad \Rightarrow \quad |x^3| < 8 \quad \Rightarrow |x| < 2.$$
Thus radius of convergence is 2 and interval of convergence is $(-2, 2)$.

Extra Practice

0.44. Example. For $\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$

(1) Find the series' radius of convergence.
(2) For what value of x does the series converges? (a) absolutely (b) conditionally.

0.45. Example. For $\sum_{n=0}^{\infty} \frac{(2x+3)^{2n+1}}{n!}$

(1) Find the series' radius of convergence.
(2) For what value of x does the series converges? (a) absolutely (b) conditionally.

0.46. Example. Examine the convergence of the series:

(1) $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \dots$
(2) $\frac{1}{\sqrt{1\cdot 2}} + \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} + \dots$
(3) $\frac{1}{1\cdot 2} + \frac{2}{3\cdot 4} + \frac{3}{5\cdot 6} + \dots$
(4) $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$
(5) $\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \dots$
(6) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \dots$
(7) $\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \dots$
(8) $\frac{1}{4\cdot 6} + \frac{\sqrt{3}}{6\cdot 8} + \frac{\sqrt{5}}{8\cdot 10} + \dots$
(9) $\frac{\sqrt{2}-1}{1} + \frac{\sqrt{3}-\sqrt{2}}{2} + \frac{\sqrt{4}-\sqrt{3}}{3} + \dots$
(10) $\frac{1\cdot 2}{3^2\cdot 4^2} + \frac{3\cdot 4}{5^2\cdot 6^2} + \frac{5\cdot 6}{7^2\cdot 8^2} + \dots$

0.47. Example. Examine the convergence of the series:

(1) $\sum_{n=1}^{\infty} \sqrt{\frac{1+2^n}{1+3^n}}$
(2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\frac{1}{n}$
(3) $\sum_{n=1}^{\infty} \frac{n+2}{n^3+1}$
(4) $\sum_{n=1}^{\infty} (3n-1)^{-1}$
(5) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$
(6) $\sum_{n=1}^{\infty} \cos\frac{1}{n}$
(7) $\sum_{n=1}^{\infty} \tan\frac{1}{n}$
(8) $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{1}{n}$
(9) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$
(10) $\sum_{n=1}^{\infty} \frac{1}{n^{(a+\frac{b}{n})}}$